TMA4275 Lifetime analysis

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The Nelson–Aalen estimator

(Reference: Section 3.1.5 in Aalen, Borgan and Gjessing, 2008)

- ⋆ Let $N = \{N(t); t \in [0, \tau]\}$ be a counting process
  - multiplicative intensity model: $\lambda(t) = \alpha(t)Y(t)$
  - $Y(t)$ is predictable process
  - want an estimator for $A(t) = \int_0^t \alpha(s)ds$

Doob–Meyer decomposition:

$X(t) = X^{\star}(t) + M(t)$

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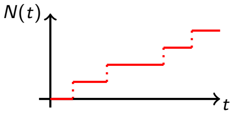

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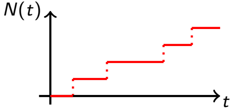

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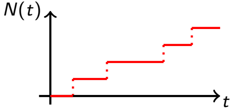

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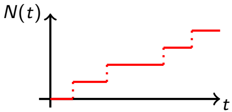

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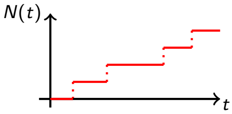

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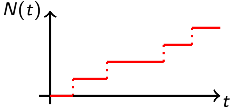

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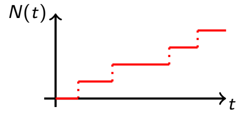

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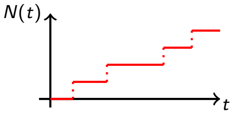

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- Then we get

$$\widehat{A}(t) = A^\star(t) + \int_0^t \frac{J(s)}{Y(s)}dM(s)$$

The Nelson–Aalen estimator (cont.)

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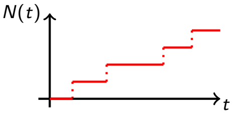

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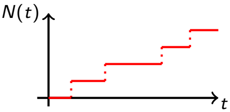

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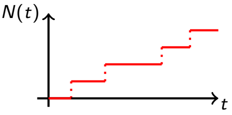

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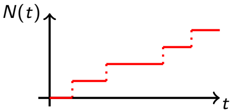

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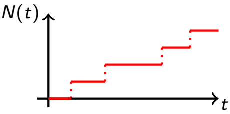

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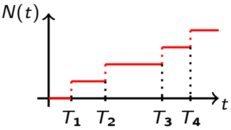

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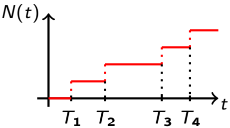

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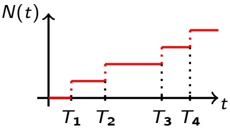

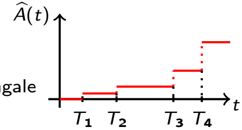

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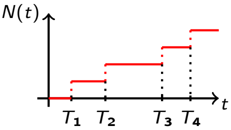

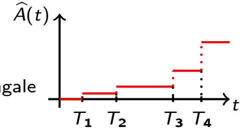

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- Next: Find (an estimator for) the variance of $\widehat{A}(t) - A^\star(t)$

The variance of $\widehat{A}(t) - A^\star(t)$

$[\int HdM](t) = \int_0^t (H(s))^2 dN(s)$

$\text{Var}[M(t)] = \text{E}[[M](t)]$

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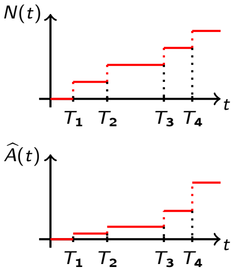

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- This gives

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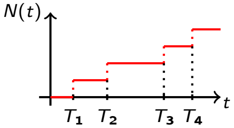

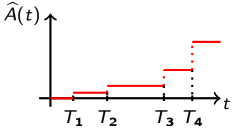

The variance of $\widehat{A}(t) - A^\star(t)$

$$[\int HdM](t) = \int_0^t (H(s))^2 dN(s)$$

$$\text{Var}[M(t)] = \text{E}[[M](t)]$$

- Recall from previous page:

$$\widehat{A}(t) - A^\star(t) = \int_0^t \frac{J(s)}{Y(s)} dM(s)$$

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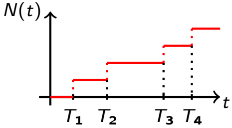
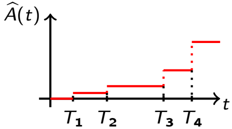
$$\left[\widehat{A} - A^\star\right](t) = \left[\int_0^t \frac{J(s)}{Y(s)} dM(s)\right] = \left[\int \frac{J}{Y} dM\right](t)$$





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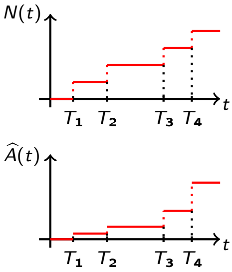

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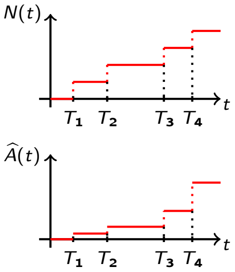

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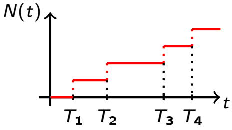
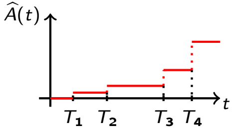
- So we get

$$\mathrm{Var}\left[\widehat{A}(t) - A^\star(t)\right] = \mathrm{E}\left[\left[\widehat{A} - A^\star\right](t)\right] = \mathrm{E}\left[\sum_{j: T_j < t} \frac{1}{Y(T_j)^2}\right]$$





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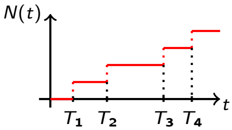

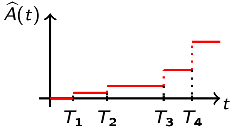

- So we get

$$\mathrm{Var}\left[\widehat{A}(t) - A^\star(t)\right] = \mathrm{E}\left[\left[\widehat{A} - A^\star\right](t)\right] = \mathrm{E}\left[\sum_{j:T_j<t} \frac{1}{Y(T_j)^2}\right]$$

- Thus: An unbiased estimator for $\mathrm{Var}\left[\widehat{A}(t) - A^\star(t)\right]$ is

$$\widehat{\sigma}^2(t) = \sum_{j:T_j<t} \frac{1}{Y(T_j)^2}$$

Summary

- We have derived
  - the Nelson–Aalen estimator for $A(t)$
  - an estimator for the variance of the Nelson–Aalen estimator

- To do this we have used
  - the Doob–Meyer decomposition (for a counting process)
  - stochastic integrals (and properties of these)
  - properties of martingales
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- ⋆ We have derived
  - the Nelson–Aalen estimator for $A(t)$
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- ⋆ To do this we have used
  - the Doob–Meyer decomposition (for a counting process)
  - stochastic integrals (and properties of these)
  - properties of martingales
  - properties of an optional variation process
- ⋆ Remaining questions:
  - what is the distribution of $\widehat{A}(t) - A^\star(t)$ (asymptotically)?
  - what do we do if we observe several events at exactly the same time?