TMA4275 Life time analysis

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Norwegian University of Science and Technology

Kaplan-Meier for a Markov chain

(Reference: Section 3.4.3 in Aalen, Borgan and Gjessing, 2008)

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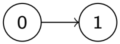

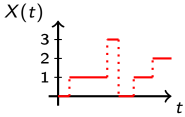

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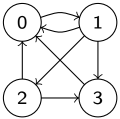

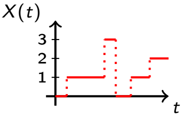

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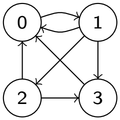

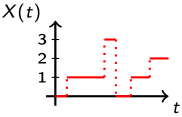

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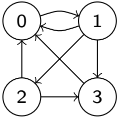

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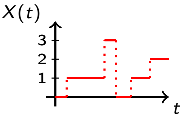

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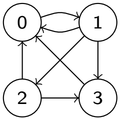

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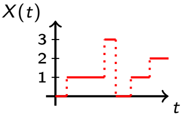

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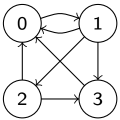

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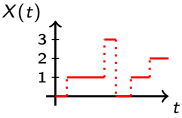

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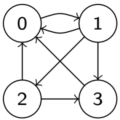

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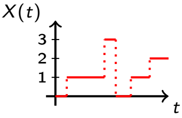

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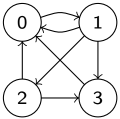

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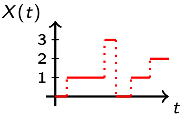

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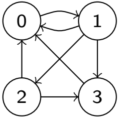

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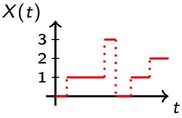

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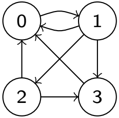

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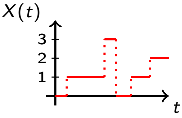

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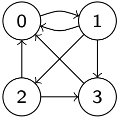

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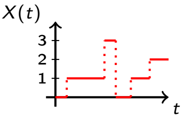

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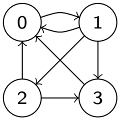

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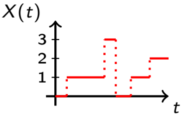

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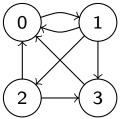

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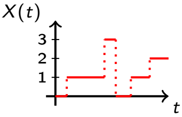

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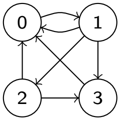

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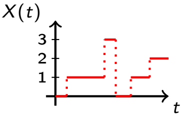

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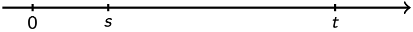
  - next: use this to find $P(s,t)$

Derivation of formula for $P(s, t)$

- ⋆ Want formula for $P(s, t)$ as a function of $\alpha(t)$



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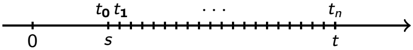

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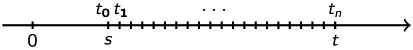

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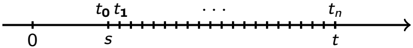

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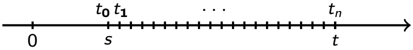

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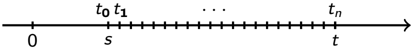

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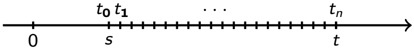

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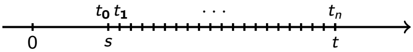

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- It can be shown that this expression is valid also when $A(t)$ is not absolutely continuous

Estimating $A(t)$ and $P(s,t)$

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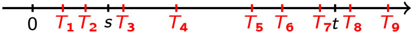

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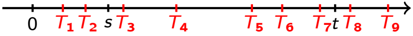

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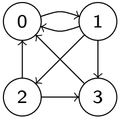

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- ⋆ Characterise the process by
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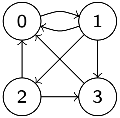

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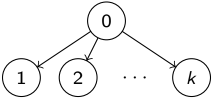
- ⋆ Note: for some Markov chain situations the expression for $\widehat{P}(s, t)$ can be simplified

competing risks



illness-death model