TMA4275 Life time analysis

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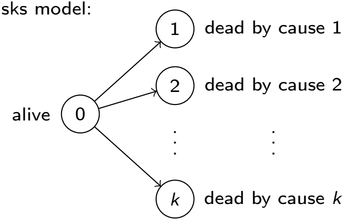
Competing risks

(Reference: Section 3.4.1 in Aalen, Borgan and Gjessing, 2008)

$$\widehat{P}(s,t) = \prod_{(s,t]} \left(\mathbb{I} + d\widehat{A}(u)\right) = \prod_{j: s < T_j \le t} \left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$$

- ⋆ Competing risks model:



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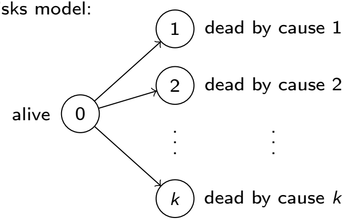

- ⋆ $P(s,t)$ has the following structure

$$P(s,t) = \begin{bmatrix} P_{00}(s,t) & P_{01}(s,t) & P_{02}(s,t) & \dots & P_{0,k}(s,t) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

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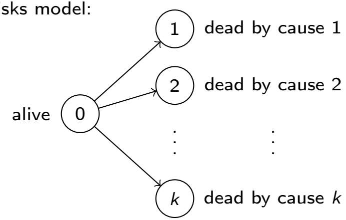

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$$P(s,t) = \begin{bmatrix} P_{\mathbf{00}}(s,t) & P_{\mathbf{01}}(s,t) & P_{\mathbf{02}}(s,t) & \dots & P_{\mathbf{0},k}(s,t) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- ⋆ Want to find formulas for $\widehat{P}_{\mathbf{00}}(s,t)$ and $\widehat{P}_{\mathbf{0}h}(s,t)$

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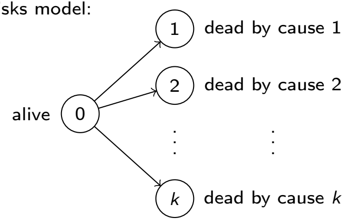


- $P(s,t)$ has the following structure

$$P(s,t) = \begin{bmatrix} P_{\mathbf{00}}(s,t) & P_{\mathbf{01}}(s,t) & P_{\mathbf{02}}(s,t) & \dots & P_{\mathbf{0},k}(s,t) \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- Want to find formulas for $\widehat{P}_{\mathbf{00}}(s,t)$ and $\widehat{P}_{\mathbf{0}h}(s,t)$
  - consider $k = 3$
  - the factor $\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$
  - product of two $\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$ factors
  - product of three $\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$ factors
  - "guess" on general formula
  - general formula can be proved by induction

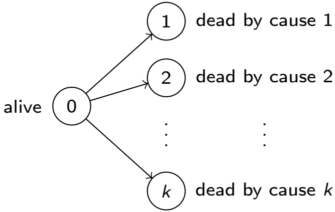
The factor $\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$

$$\widehat{P}(s,t) = \prod_{(s,t]} \left(\mathbb{I} + d\widehat{A}(u)\right) = \prod_{j: s < T_j \leq t} \left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$$

$$\widehat{A}_{gh}(t) = \int_0^t \frac{J_g(s)}{Y_g(s)}\, dN_{gh}(s) = \sum_{j: T_j^{gh} \leq t} \frac{1}{Y_g(T_j^{gh})}$$

$$\widehat{A}_{gg}(t) = -\sum_{h \neq g} \widehat{A}_{gh}(t)$$

⋆ Competing risks model



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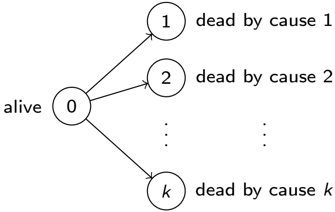

⋆ Recall: Observe n individuals following the competing risks model

The factor $\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$

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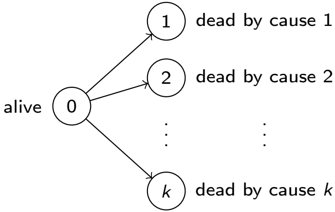
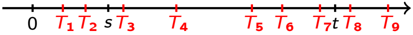


⋆ Recall: Observe n individuals following the competing risks model

⋆ Notation: $0 < T_1 < T_2 < \dots$



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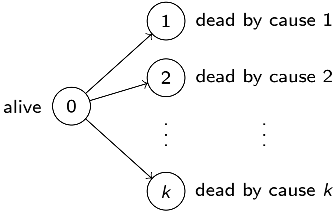

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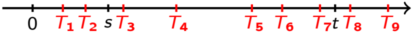

- ⋆ For $k = 3$, if we at time T_j observe a transition from state 0 to state 2 we have

$$\Delta\widehat{A}(T_j) = \begin{bmatrix} -\frac{1}{Y_0(T_j)} & 0 & \frac{1}{Y_0(T_j)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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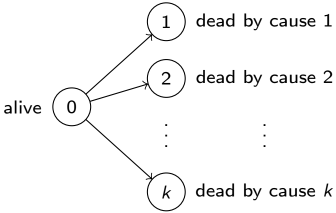

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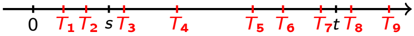

- ⋆ For $k = 3$, if we at time T_j observe a transition from state 0 to some other state, we have

$$\Delta\widehat{A}(T_j) = \begin{bmatrix} -\frac{1}{Y_0(T_j)} & \frac{\Delta N_{01}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{02}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{03}(T_j)}{Y_0(T_j)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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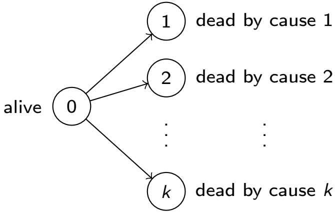

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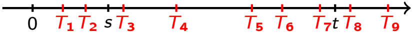

- ⋆ For $k = 3$, if we at time T_j observe a transition from state 0 to some other state, we have

$$\mathbb{I} + \Delta\widehat{A}(T_j) = \begin{bmatrix} 1 - \frac{1}{Y_0(T_j)} & \frac{\Delta N_{01}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{02}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{03}(T_j)}{Y_0(T_j)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of two factors

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right)$$

$$= \begin{bmatrix} 1 - \frac{1}{Y_0(T_j)} & \frac{\Delta N_{01}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{02}(T_j)}{Y_0(T_j)} & \frac{\Delta N_{03}(T_j)}{Y_0(T_j)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 - \frac{1}{Y_0(T_k)} & \frac{\Delta N_{01}(T_k)}{Y_0(T_k)} & \frac{\Delta N_{02}(T_k)}{Y_0(T_k)} & \frac{\Delta N_{03}(T_k)}{Y_0(T_k)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} a(T_j) & b_1(T_j) & b_2(T_j) & b_3(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a(T_k) & b_1(T_k) & b_2(T_k) & b_3(T_k) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of two factors

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right)$$

$$= \begin{bmatrix} 1 - \frac{1}{Y_{\mathbf{0}}(T_j)} & \frac{\Delta N_{\mathbf{01}}(T_j)}{Y_{\mathbf{0}}(T_j)} & \frac{\Delta N_{\mathbf{02}}(T_j)}{Y_{\mathbf{0}}(T_j)} & \frac{\Delta N_{\mathbf{03}}(T_j)}{Y_{\mathbf{0}}(T_j)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 - \frac{1}{Y_{\mathbf{0}}(T_k)} & \frac{\Delta N_{\mathbf{01}}(T_k)}{Y_{\mathbf{0}}(T_k)} & \frac{\Delta N_{\mathbf{02}}(T_k)}{Y_{\mathbf{0}}(T_k)} & \frac{\Delta N_{\mathbf{03}}(T_k)}{Y_{\mathbf{0}}(T_k)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a(T_j) & b_{\mathbf{1}}(T_j) & b_{\mathbf{2}}(T_j) & b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a(T_k) & b_{\mathbf{1}}(T_k) & b_{\mathbf{2}}(T_k) & b_{\mathbf{3}}(T_k) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a(T_j)a(T_k) & a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Product of three factors

$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k) & a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 - \frac{1}{Y_{\mathbf{0}}(T_l)} & \frac{\Delta N_{\mathbf{01}}(T_l)}{Y_{\mathbf{0}}(T_l)} & \frac{\Delta N_{\mathbf{02}}(T_l)}{Y_{\mathbf{0}}(T_l)} & \frac{\Delta N_{\mathbf{03}}(T_l)}{Y_{\mathbf{0}}(T_l)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} a(T_j)a(T_k) & a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$= \left[\begin{matrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right]$$

Product of three factors

$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$

$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

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Product of three factors

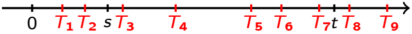
$a(T_j) = 1 - \frac{1}{Y_0(T_j)}$

$b_h(T_j) = \frac{\Delta N_{0h}(T_j)}{Y_0(T_j)}$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_1(T_l) + a(T_j)b_1(T_k) + b_1(T_j) & a(T_j)a(T_k)b_2(T_l) + a(T_j)b_2(T_k) + b_2(T_j) & a(T_j)a(T_k)b_3(T_l) + a(T_j)b_3(T_k) + b_3(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋆ From this we see that



$$\widehat{P}_{00}(s,t) = \left[\prod_{j:s<T_j\le t} \left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{11}$$

Product of three factors

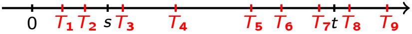
$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \left[\begin{array}{cccc} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

- ⋆ From this we see that



$$\widehat{P}_{\mathbf{00}}(s,t) = \left[\prod_{j:s<T_j\leq t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{11}} = \prod_{j:s<T_j\leq t} a(T_j)$$

Product of three factors

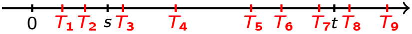
$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ⋆ From this we see that



$$\widehat{P}_{\mathbf{00}}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{11}} = \prod_{j:s<T_j\le t} a(T_j) = \prod_{j:s<T_j\le t}\left(1 - \frac{1}{Y_{\mathbf{0}}(T_j)}\right)$$

Product of three factors

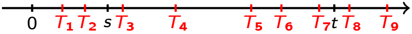
$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋆ From this we see that



$$\widehat{P}_{\mathbf{00}}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{11}} = \prod_{j:s<T_j\le t} a(T_j) = \prod_{j:s<T_j\le t}\left(1 - \frac{1}{Y_{\mathbf{0}}(T_j)}\right)$$

$$\widehat{P}_{\mathbf{0}h}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{1},h+\mathbf{1}}$$

Product of three factors

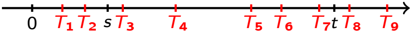
$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋆ From this we see that



$$\widehat{P}_{\mathbf{00}}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{11}} = \prod_{j:s<T_j\le t} a(T_j) = \prod_{j:s<T_j\le t}\left(1 - \frac{1}{Y_{\mathbf{0}}(T_j)}\right)$$

$$\widehat{P}_{\mathbf{0}h}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{1},h+\mathbf{1}} = \sum_{j:s<T_j\le t}\left[\left(\prod_{k:s<T_k<T_j} a(T_k)\right) b_h(T_j)\right]$$

Product of three factors

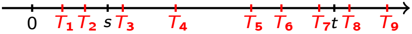
$$a(T_j) = 1 - \frac{1}{Y_0(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{0h}(T_j)}{Y_0(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_1(T_l) + a(T_j)b_1(T_k) + b_1(T_j) & a(T_j)a(T_k)b_2(T_l) + a(T_j)b_2(T_k) + b_2(T_j) & a(T_j)a(T_k)b_3(T_l) + a(T_j)b_3(T_k) + b_3(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋆ From this we see that



$$\widehat{P}_{00}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{11} = \prod_{j:s<T_j\le t} a(T_j) = \prod_{j:s<T_j\le t}\left(1 - \frac{1}{Y_0(T_j)}\right)$$

$$\widehat{P}_{0h}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{1,h+1} = \sum_{j:s<T_j\le t}\left[\left(\prod_{k:s<T_k<T_j} a(T_k)\right) b_h(T_j)\right]$$

$$= \sum_{j:s<T_j\le t} \widehat{P}_{00}(s, T_j-) \cdot \frac{\Delta N_{0h}(T_j)}{Y_0(T_j)}$$

Product of three factors

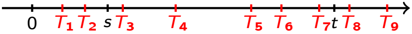
$$a(T_j) = 1 - \frac{1}{Y_{\mathbf{0}}(T_j)}$$

$$b_h(T_j) = \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

$$\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_k)\right) \times \left(\mathbb{I} + \Delta\widehat{A}(T_l)\right)$$

$$= \begin{bmatrix} a(T_j)a(T_k)a(T_l) & a(T_j)a(T_k)b_{\mathbf{1}}(T_l) + a(T_j)b_{\mathbf{1}}(T_k) + b_{\mathbf{1}}(T_j) & a(T_j)a(T_k)b_{\mathbf{2}}(T_l) + a(T_j)b_{\mathbf{2}}(T_k) + b_{\mathbf{2}}(T_j) & a(T_j)a(T_k)b_{\mathbf{3}}(T_l) + a(T_j)b_{\mathbf{3}}(T_k) + b_{\mathbf{3}}(T_j) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋆ From this we see that



$$\widehat{P}_{\mathbf{00}}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{11}} = \prod_{j:s<T_j\le t} a(T_j) = \prod_{j:s<T_j\le t}\left(1 - \frac{1}{Y_{\mathbf{0}}(T_j)}\right)$$

$$\widehat{P}_{\mathbf{0}h}(s,t) = \left[\prod_{j:s<T_j\le t}\left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)\right]_{\mathbf{1},h+\mathbf{1}} = \sum_{j:s<T_j\le t}\left[\left(\prod_{k:s<T_k<T_j} a(T_k)\right) b_h(T_j)\right]$$

$$= \sum_{j:s<T_j\le t} \widehat{P}_{\mathbf{00}}(s, T_j-) \cdot \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)} = \sum_{j:s<T_j\le t} \widehat{P}_{\mathbf{00}}(s, T_{j-\mathbf{1}}) \cdot \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$

Proof for the formulas

- ⋆ Want to proof the formulas

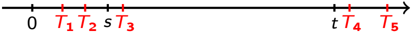
$$\widehat{P}_{00}(s,t) = \prod_{j: s < T_j \le t} \left(1 - \frac{1}{Y_0(T_j)}\right)$$

$$\widehat{P}_{0h}(s,t) = \sum_{j: s < T_j \le t} \widehat{P}_{00}(s, T_{j-1}) \cdot \frac{\Delta N_{0h}(T_j)}{Y_0(T_j)}$$

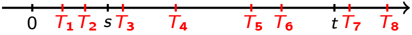
- ⋆ Given

$$\widehat{P}(s,t) = \prod_{j: s < T_j \le t} \left(\mathbb{I} + \Delta \widehat{A}(T_j)\right)$$

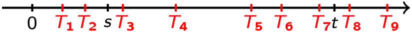
- ⋆ Proof by induction
  - – proof formulas when only one observed transition in $(s,t]$



  - – assume formulas correct for k observed transition in $(s,t]$,

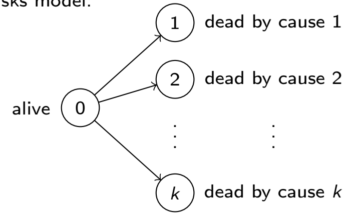


and show that then it is also correct with $k+1$ observed transitions in $(s,t]$



Summary

- Competing risks model:



- Used Kaplan–Meier estimator for general Markov chain

$$\widehat{P}(s,t) = \prod_{(s,t]} \left(\mathbb{I} + d\widehat{A}(u)\right) = \prod_{j: s < T_j \le t} \left(\mathbb{I} + \Delta\widehat{A}(T_j)\right)$$

- Showed that

$$\widehat{P}_{\mathbf{00}}(s,t) = \prod_{j: s < T_j \le t} \left(1 - \frac{1}{Y_{\mathbf{0}}(T_j)}\right)$$

$$\widehat{P}_{\mathbf{0}h}(s,t) = \sum_{j: s < T_j \le t} \widehat{P}_{\mathbf{00}}(s, T_{j-\mathbf{1}}) \cdot \frac{\Delta N_{\mathbf{0}h}(T_j)}{Y_{\mathbf{0}}(T_j)}$$