TMA4275 Lifetime analysis

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How to describe the distribution of a life time?

(Reference: Section 1.1.2 in Aalen, Borgan and Gjessing, 2008)

- T: Life time, continuous stochastic variable
- How to characterise its distribution?

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$$P(a < T \leq b) = \int_a^b f(t)dt$$

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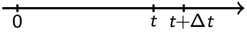

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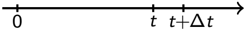

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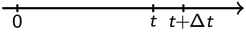

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- *Next:*
  - *relations between* $\alpha(t)/A(t)$ *and* $S(t)$

Relation between $\alpha(t)/A(t)$ and $S(t)$

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\int_0^t \alpha(s)ds &= -[\ln S(t)]_0^t \\
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\int_0^t \alpha(s)ds &= - [\ln S(t)]_0^t \\
A(t) &= - [\ln S(t) - \ln S(0)] \\
S(t) &= e^{-A(t)}
\end{aligned}$$

Example: The exponential distribution

$$F(t) = \int_0^t f(s)ds$$
$$S(t) = 1 - F(t)$$
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$$A(t) = \int_0^t \lambda ds = [\lambda s]_0^t = \lambda t, t > 0$$

Summary

- ⋆ How to describe the distribution of a life time T?
  - when T is a continuous stochastic variable
- ⋆ We are from before familiar with
  - $f(t)$
  - $F(t)$
- ⋆ We have now also introduced
  - $S(t)$
  - $\alpha(t)$
  - $A(t)$
- ⋆ We have found relations between these five functions