TMA4275 Lifetime analysis

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Counting processes

(Reference: Section 1.4 in Aalen, Borgan and Gjessing, 2008)

- ⋆ Counting process
  - consider a particular type of event happening on a time axis
  - assume at most one event (of this type) happens in a short time interval $(t, t + dt]$
  - $N(t)$: number of events of this type from type 0 up to (and including) time t
  - increment: $dN(t) = N(t + dt) - N(t)$
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- $M(t)$ is a martingale

One (uncensored) survival time

- ⋆ One survival time: T (continuous stochastic variable)
- ⋆ Hazard rate: $\alpha(t)$

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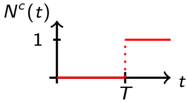

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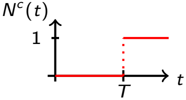

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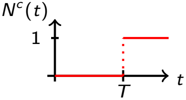

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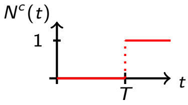

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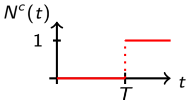

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- ⋆ *Next: n independent (uncensored) survival times $T_1, \ldots, T_n$*

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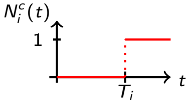

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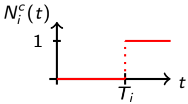

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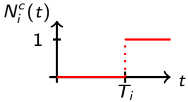

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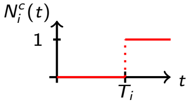

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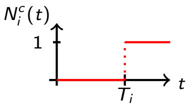

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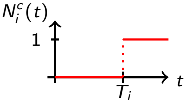

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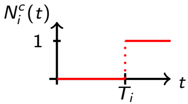

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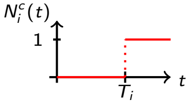

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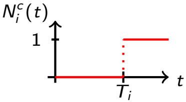

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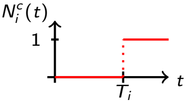

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$$\lambda^c(t) = \sum_{i=1}^{n} \alpha_i(t) I(T_i \geq t)$$

- If $\alpha_1(t) = \ldots = \alpha_n(t) = \alpha(t)$ we get

$$\lambda^c(t) = \alpha(t) Y^c(t)$$

where

$$Y^c(t) = \sum_{i=1}^{n} I(T_i \geq t) : \text{ the number of individuals at risk just before time } t$$

Independent censoring — one survival time

- ⋆ Survival time: T (continuous stochastic variables)
- ⋆ Hazard rate: $\alpha(t)$
- ⋆ We observe $(\widetilde{T}, D)$, where
  - $\widetilde{T}$: right-censored survival time
  - D: censoring indicator

$$D = \begin{cases} 1 & \text{if } \widetilde{T} = T, \\ 0 & \text{if } \widetilde{T} < T \end{cases}$$

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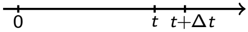
$$P(t \leq \widetilde{T} < t + dt | \widetilde{T} \geq t, \text{past}) = P(t \leq T < t + dt | T \geq t)$$



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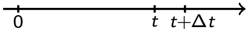

- Intensity process for $N(t)$

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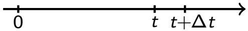

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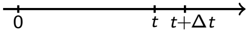

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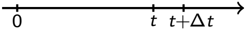

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- ⋆ *Next: n right-censored survival times $\widetilde{T}_1, \ldots, \widetilde{T}_n$*

Independent censoring — several survival times

- ⋆ Survival times: $T_1, \ldots, T_n$ (independent continuous stochastic variables)
- ⋆ Hazard rates: $\alpha_1(t), \ldots, \alpha_n(t)$
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- ⋆ Aggregated counting process

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Summary

- Have defined (informally)
  - counting process
  - intensity process of a counting process, $\lambda(t)$
  - independent censoring

- Have found formulas for $\lambda(t)$ in the case of
  - one uncensored survival time
  - several uncensored survival times
  - one censored survival time (assuming independent censoring)
  - several censored survival times (assuming independent censoring)

- Multiplicative intensity process

$$\lambda(t) = \alpha(t)Y(t)$$